J. Fluid Mech. (1975), vol. 72, part 2, pp. 353–362
Printed in Great Britain

On an oscillatory point force in a rotating stratified fluid

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(Received 29 November 1974)

The forced oscillations due to a point forcing effect in an infinite or contained, inviscid, incompressible, rotating, stratified fluid are investigated taking into account the density variation in the inertia terms in the linearized equations of motion. The solutions are obtained in closed form using generalized Fourier transforms. Solutions are presented for a medium bounded by a finite cylinder when the oscillatory forcing effect is acting at a point on the axis of the cylinder. In both the unbounded and bounded case, there exist characteristic cones emanating from the point of application of the force on which either the pressure or its derivatives are discontinuous. The perfect resonance existing at certain frequencies in an unbounded or bounded homogeneous fluid is avoided in the case of a confined stratified fluid.

1. Introduction

The motion due to oscillations of a small disk placed in the interior of a cylindrical container has been studied by Görtler (1957). He predicted the existence of discontinuous surfaces similar to Mach cones in compressible aerodynamics when the forcing frequency ω is less than the natural rotational frequency 2Ω. These results were experimentally verified by Oser (1958). Further, it may be seen that the cones of discontinuity continue by reflexion off the lateral wall of the cylinder. In the above investigations only the theory for an unbounded domain was considered. Reynolds (1962*a*, *b*) has analysed forced oscillations in rotating fluids using an initial-value formulation and has shown that for the case of rapid forcing the motion set up after a long time will be everywhere in phase with the exciting motion, whereas, for slow forcing, travelling waves will be set up in the fluid, and, in addition to the motion in phase with the excitation, an out-of-phase component will also exist.

Forced oscillations of a contained inviscid rotating fluid in a finite cylinder have been considered by Baines (1967). The motion was induced by forcing simultaneous identical and axisymmetric time-harmonic deformations of the plane ends of the cylinder. The solution was obtained by using Laplace transform techniques and consists of an infinite set of inertial modes in addition to motion which oscillates with the forcing frequency. Further, it was concluded that the inviscid form of steady forced motion should only be approached in a viscous medium. These theoretical predictions are in good agreement with the experimental findings of McEwan (1971). Devanathan & Ramachandra Rao (1973) have extended the above problem to include the effects of stratification; some new features appear when the density variation is included in the inertia terms.

The aim of the present investigation is to study the oscillations forced in a contained rotating stratified fluid by an oscillatory point force acting at a point on the axis of rotation. It is well known that small obstacles or bodies can be replaced formally by point forces, which are easier to deal with. The slow motion of a sphere in a rotating viscous fluid has been studied by Childress (1964) by replacing the sphere by a point force equal to the Stokes drag on the sphere. Janowitz (1968), in his study of wakes in stratified fluids, replaced the boundary conditions on the body by linearized momentum integral equations, which in turn were replaced by a product of two Dirac delta functions. The governing equation for the pressure is derived in §2. In §3 the solutions for a point force in an unbounded fluid are presented in closed form. These solutions are different from those obtained by Sarma & Naidu (1972) in the hyperbolic case as they did not make use of a radiation condition, which is necessary to pick the correct solution. In §4, the solutions for a point force acting at a point on the axis of a finite cylinder containing the rotating stratified fluid are given in terms of infinite series. By analysing the series by a method similar to that given by Baines (1967), it is shown that the derivatives of the pressure become discontinuous (the series diverge) along certain cones and that these cones continue by reflexion at the lateral surface of the cylinder.

2. Formulation

We consider a fluid rotating about the Cartesian z axis with an angular velocity 2Ω, the rotation axis being assumed to be antiparallel to gravity. The equations of motion of an inviscid incompressible fluid in a rotating frame of reference are

$$\rho'[\partial \mathbf{q}/\partial t + \mathbf{q}\,.\,\nabla \mathbf{q} + 2\Omega \hat{\mathbf{z}} \times \mathbf{q} + \Omega^2 \hat{\mathbf{z}} \times \hat{\mathbf{z}} \times \mathbf{r} - \mathbf{X}'] = -\nabla p' - \rho' g \hat{\mathbf{z}}, \tag{1}$$

where ρ' is the density, $\mathbf{q}$ is the velocity vector, p' is the pressure, $\hat{\mathbf{z}}$ is a unit vector in the z direction, g is the acceleration due to gravity, $\mathbf{r}$ is the position vector and $\mathbf{X}'$ is the applied external force.

The condition that the fluid be incompressible gives

$$\partial \rho'/\partial t + (\mathbf{q}\,.\,\nabla)\rho' = 0 \tag{2}$$

and the equation of continuity is

$$\nabla\,.\,\mathbf{q} = 0. \tag{3}$$

The stratification of the undisturbed fluid is taken as $\rho_0(z) = \rho_0' e^{-\beta z}$ for all finite z, and for $z \to \infty$ we assume that ρ_0 approaches slowly, smoothly and monotonically some positive value. For this stratification, the Brunt–Väisälä frequency $N = [(-g/\rho_0)(d\rho_0/dz)]^{\frac{1}{2}}$ remains constant throughout the fluid. Linearizing (1)–(3) under the assumption that u, v, w, p and ρ, the perturbation velocity components, pressure and density, are small and taking into consideration the density variation in the inertial and applied external forces, we get

$$\rho_0[\partial u/\partial t - 2\Omega v + \Omega^2 x - X'] = -\partial p/\partial x, \tag{4}$$

$$\rho_0[\partial v/\partial t + 2\Omega u + \Omega^2 y - Y'] = -\partial p/\partial y, \tag{5}$$

$$\rho_0[\partial w/\partial t - Z'] = -\partial p/\partial z - \rho g, \tag{6}$$

$$\frac{\partial \rho}{\partial t} + w\frac{d\rho_0}{dz} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{7), (8}$$

where $\mathbf{X}' = (X', Y', Z')$. Eliminating u, v and w from (4)–(6) using (7) and (8), the equation governing the pressure is obtained as

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{4\Omega^2 - \omega^2}{N^2 - \omega^2}\left(\frac{\partial^2 P}{\partial z^2} + \beta\frac{\partial P}{\partial z}\right) = \rho_0\left[\left(\frac{\partial}{\partial x} - \frac{2\Omega}{i\omega}\frac{\partial}{\partial y}\right)X + \left(\frac{\partial}{\partial y} + \frac{2\Omega}{i\omega}\frac{\partial}{\partial x}\right)Y + \frac{4\Omega^2 - \omega^2}{N^2 - \omega^2}\frac{\partial Z}{\partial z}\right], \tag{9}$$

where

$$p = Pe^{i\omega t}, \quad (X', Y', Z') = (X, Y, Z)\,e^{i\omega t}. \tag{10}$$

The term $\beta(\partial P/\partial z)$ on the left-hand side of (9) will be absent and ρ_0 on the right-hand side of (9) will be a constant if the density variation is not included in the inertia terms, i.e. when the Boussinesq approximation, in which the density variation is considered only in buoyancy forces, is made. We call a fluid Boussinesq or non-Boussinesq according to whether the Boussinesq approximation holds or not. We observe that the equations governing the flow of a Boussinesq rotating fluid and a homogeneous rotating fluid are very similar and thus their solutions are also very similar. But when density variation is considered in the inertial forces, that is, for a non-Boussinesq fluid, some new features are observed (Sarma & Krishna 1972; Devanathan & Ramachandra Rao 1973).

Now let us consider the applied external force be a point force acting at a point $(0, 0, z_0)$ on the z axis and given by

$$\mathbf{X}' = e^{i\omega t}(L', M', N')\,\delta(x)\,\delta(y)\,\delta(z - z_0), \tag{11}$$

where L', M' and N' are constant and $\delta(x)$ is a Dirac delta function. Substituting (11) in (9), we get

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{4\Omega^2 - \omega^2}{N^2 - \omega^2}\left(\frac{\partial^2 P}{\partial z^2} + \beta\frac{\partial P}{\partial z}\right) = \left[L'\left(\frac{\partial}{\partial x} - \frac{2\Omega}{i\omega}\frac{\partial}{\partial y}\right) + M'\left(\frac{\partial}{\partial y} + \frac{2\Omega}{i\omega}\frac{\partial}{\partial x}\right) + N'\frac{4\Omega^2 - \omega^2}{N^2 - \omega^2}\left(\frac{\partial}{\partial z} + \beta\right)\right]\rho_0\,\delta(x)\,\delta(y)\,\delta(z - z_0). \tag{12}$$

Equations (9) and (12) are elliptic, hyperbolic or parabolic in the spatial coordinates according as $(4\Omega^2 - \omega^2)/(N^2 - \omega^2)$ is positive, negative or zero (infinite). In other words the equations are hyperbolic if $2\Omega < \omega < N$ or $N < \omega < 2\Omega$, elliptic if $2\Omega, N > \omega$ or $2\Omega, N < \omega$ and parabolic if $2\Omega = \omega$ or $N = \omega$.

3. The solutions for a point force in an unbounded fluid

The problem here is to solve (12) subject to the condition that the pressure vanishes at infinity. Substituting

$$P = \left[L'\left(\frac{\partial}{\partial x} - \frac{2\Omega}{i\omega}\frac{\partial}{\partial y}\right) + M'\left(\frac{\partial}{\partial y} + \frac{2\Omega}{i\omega}\frac{\partial}{\partial x}\right) + N'\frac{4\Omega^2-\omega^2}{N^2-\omega^2}\left(\frac{\partial}{\partial z}+\beta\right)\right]P_0 \quad (13)$$

in (12) and writing it in cylindrical polar co-ordinates (r, θ, z), we obtain the equation governing P_0 as

$$\frac{\partial^2 P_0}{\partial r^2} + \frac{1}{r}\frac{\partial P_0}{\partial r} + \frac{4\Omega^2-\omega^2}{N^2-\omega^2}\left(\frac{\partial^2 P_0}{\partial z^2} + \beta\frac{\partial P_0}{\partial z}\right) = \frac{\rho_0(z)}{2\pi r}\delta(r)\,\delta(z-z_0). \quad (14)$$

The fact that the solution for P_0 will be axisymmetric has been used in writing down (14). By taking a generalized Fourier transform with respect to z, (14) becomes

$$\frac{d^2\bar{P}_0}{dr^2} + \frac{1}{r}\frac{d\bar{P}_0}{dr} - \mu^2\gamma^2\bar{P}_0 = \frac{\rho_0(z_0)}{2\pi r}e^{i\alpha z_0}\delta(r) \quad \text{(elliptic case)}, \quad (15)$$

$$\frac{d^2\bar{P}_0}{dr^2} + \frac{1}{r}\frac{d\bar{P}_0}{dr} + \lambda^2\gamma^2\bar{P}_0 = \frac{\rho_0(z_0)\,e^{i\alpha z_0}}{2\pi r}\delta(r) \quad \text{(hyperbolic case)}, \quad (16)$$

where
$$\bar{P}_0 = \int_{-\infty}^{\infty} P_0 e^{i\alpha z}dz, \quad \gamma^2 = \alpha^2 + i\alpha\beta$$

and
$$(4\Omega^2-\omega^2)/(N^2-\omega^2) = \mu^2 = -\lambda^2, \quad \lambda, \mu > 0. \quad (17)$$

The solution of (15) which remains finite at infinity is

$$\bar{P}_0 = (\rho_0(z_0)/4\pi)\,e^{-i\alpha z_0}K_0(\mu\gamma r), \quad (18)$$

where $K_0(\mu\gamma r)$ is a modified Bessel function of the second kind, and the solution of (16) satisfying the Sommerfeld radiation condition is

$$\bar{P}_0 = \tfrac{1}{8}i\rho_0(z_0)\,e^{i\alpha z_0}H_0^{(2)}(\lambda\gamma r), \quad (19)$$

where $H_0^{(2)}(\lambda\gamma r)$ is a Hankel function of the second kind. Inverting the Fourier transform and making use of the results given in Erdélyi *et al.* (1954, p. 56, formulae 43, 42), the solution for P_0 in the elliptic case is

$$P_0 = \frac{\rho_0(z_0)\exp\{-\frac{1}{2}\beta[z_1 + (z_1^2+\mu^2r^2)]\}}{8\pi(z_1^2+\mu^2r^2)^{\frac{1}{2}}}, \quad |z_1| > 0, \quad (20)$$

and in the hyperbolic case is

$$P_0 = \begin{cases} \dfrac{-\rho_0(z_0)\exp\{-\frac{1}{2}\beta[z_1+(z_1^2-\lambda^2r^2)]\}}{8\pi(z_1^2-\lambda^2r^2)^{\frac{1}{2}}}, & |z_1| > \lambda r, \quad (21) \\[2ex] \dfrac{i\rho_0(z_0)\exp\{-\frac{1}{2}\beta[z_1+i(\lambda^2r^2-z_1^2)^{\frac{1}{2}}]\}}{8\pi(\lambda^2r^2-z_1^2)^{\frac{1}{2}}}, & |z_1| < \lambda r, \quad (22) \end{cases}$$

where $z_1 = z - z_0$. If the point force is acting at the origin ($z_0 = 0$), then these solutions (20)–(22) coincide with the solutions obtained by Ramachandra Rao (1973) for a mass source except for a constant multiplying factor. Thus the solutions for a point force can be obtained from the solutions for a mass source by the process of differentiation given in (13). This leads to the concept of multipoles for rotating stratified fluids; the corresponding concept of multipoles for homogeneous rotating fluids is given by Ramachandra Rao (1972). The solutions for the pressure for a Boussinesq rotating fluid are given by

$$P_0 = \rho_0'/8\pi(z_1^2 + \mu^2 r^2)^{\frac{1}{2}} \quad \text{(elliptic case)} \tag{23}$$

and

$$P_0 = \begin{cases} -\rho_0'/8\pi(z_1^2 - \lambda^2 r^2)^{\frac{1}{2}}, & |z_1| > \lambda r \\ i\rho_0'/8\pi(\lambda^2 r^2 - z_1^2)^{\frac{1}{2}}, & |z_1| < \lambda r \end{cases} \quad \text{(hyperbolic case).} \tag{24}$$

From the expressions for P_0 for Boussinesq and non-Boussinesq fluids it is clear that the exponential damping of the non-wavy disturbance in the elliptic case and the presence of waves in the hyperbolic case are essentially the new features for a non-Boussinesq fluid of the type we have considered. However, the solutions for a Boussinesq rotating fluid and a homogeneous rotating fluid are similar and the solutions for P_0 in the latter case may be obtained from (23) and (24) by putting $\mu^2 = (\omega^2 - 4\Omega^2)/\omega^2$ and $\lambda^2 = (4\Omega^2 - \omega^2)/\omega^2$.

Using the process of differentiation given in (13), the solution for the pressure when the point force is acting in the z direction ($L' = M' = 0, N' \neq 0$) in the elliptic case is

$$p = -\frac{N' e^{i\omega t}\rho_0(z_0)\,\mu^2\{z_1 + \frac{1}{2}\beta[z_1(z_1^2+\mu^2 r^2)^{\frac{1}{2}} - z_1^2 - \mu^2 r^2]\}}{8\pi(z_1^2+\mu^2 r^2)^{\frac{3}{2}}} \exp\{-\tfrac{1}{2}\beta[z_1 + (z_1^2+\mu^2 r^2)^{\frac{1}{2}}]\}, \tag{25}$$

and in the hyperbolic case is

$$p = -\frac{N' e^{i\omega t}\rho_0(z_0)\,\lambda^2\{z_1 + \frac{1}{2}\beta[z_1(z_1^2-\lambda^2 r^2)^{\frac{1}{2}} - z_1^2 + \lambda^2 r^2]\}}{8\pi(z_1^2-\lambda^2 r^2)^{\frac{3}{2}}} \times \exp\{-\tfrac{1}{2}\beta[z_1 + (z_1^2-\lambda^2 r^2)^{\frac{1}{2}}]\}, \quad |z_1| > \lambda r, \tag{26}$$

$$p = -\frac{iN' e^{i\omega t}\rho_0(z_0)\,\lambda^2\{z_1 + \frac{1}{2}\beta[iz_1(\lambda^2 r^2 - z_1^2)^{\frac{1}{2}} - z_1^2 + \lambda^2 r^2]\}}{8\pi(\lambda^2 r^2 - z_1^2)^{\frac{3}{2}}} \times \exp\{-\tfrac{1}{2}\beta[z_1 + i(\lambda^2 r^2 - z_1^2)^{\frac{1}{2}}]\}, \quad |z_1| < \lambda r. \tag{27}$$

From the expressions for p in (25)–(27), we observe that the flow pattern is axisymmetric, whereas for a force acting in either the x or the y direction the flow pattern will not remain axisymmetric, as may easily be seen from the process of differentiation given in (13). Further, in the hyperbolic case the pressure becomes discontinuous, actually becoming infinite, on the cone $z_1^2 = \lambda^2 r^2$ which emanates from the point of application of the force.

The solutions for the pressure p enable us to obtain the velocity components

v_r, v_θ and v_z from the axisymmetric equations of motion in cylindrical polar co-ordinates:

$$v_r = -\frac{1}{\rho_0 r(4\Omega^2 - N^2)}\int^r r\frac{\partial}{\partial t}\left(\nabla^2 p + \beta\frac{\partial p}{\partial z}\right)dr, \tag{28}$$

$$v_\theta = \frac{2\Omega}{\rho_0 r(4\Omega^2 - N^2)}\int^r r\left(\nabla^2 p + \beta\frac{\partial p}{\partial z}\right)dr, \tag{29}$$

$$v_z = \frac{1}{4\Omega^2 - N^2}\int^z \frac{1}{\rho_0}\frac{\partial}{\partial t}\left(\nabla^2 p + \beta\frac{\partial p}{\partial z}\right)dz, \tag{30}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Hence the velocities are effectively linear functions of the first derivatives of the pressure. The expressions for the velocities in (28)–(30) coincide with those obtained by Baines (1967) for a homogeneous rotating fluid in the limit $\beta \to 0$. Further, it may be observed that when $\omega \to 2\Omega$ the pressures given in (25)–(27) vanish everywhere except on the cone of discontinuity, which becomes a plane perpendicular to the axis of rotation passing through z_0, across which p is discontinuous. When $\omega \to N$, the pressure becomes infinite and the system resonates.

4. The solutions for a point force in fluid bounded by a finite circular cylinder

Consider the fluid to be contained in a finite cylinder of length $2l$ and radius a and let the point force be acting at the point $(0, 0, z_0)$ in the z direction

$$(L' = M' = 0, \quad N' \neq 0).$$

The governing equation for the pressure from (12) is

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{4\Omega^2 - \omega^2}{N^2 - \omega^2}\left(\frac{\partial^2 P}{\partial z^2} + \beta\frac{\partial P}{\partial z}\right) = \frac{N'(4\Omega^2 - \omega^2)\rho_0(z)}{2\pi(N^2 - \omega^2)r}\delta(r)\frac{d\delta(z - z_0)}{dz}. \tag{31}$$

The boundary conditions are

$$v_r = 0 \quad \text{on} \quad r = a, \qquad v_z = 0 \quad \text{on} \quad z = \pm l. \tag{32a, b}$$

Equations (32) in terms of P are

$$\partial P/\partial r = 0 \quad \text{on} \quad r = a, \qquad \partial P/\partial z = 0 \quad \text{on} \quad z = \pm l. \tag{33a, b}$$

By applying to (31) a finite Hankel transform defined by

$$\tilde{P}(\xi_n, z) = \int_0^a rP(r, z)\, J_0(\xi_n r)\, dr, \tag{34}$$

where the ξ_n are the positive roots taken in order of $J_0'(\xi a) = 0$, and making use of boundary condition (33a), we get

$$\frac{d^2\tilde{P}}{dz^2} + \beta\frac{d\tilde{P}}{dz} - \frac{N^2 - \omega^2}{4\Omega^2 - \omega^2}\xi_n^2\tilde{P} = \frac{N'\rho_0(z)}{4\pi}\frac{d\delta(z - z_0)}{dz}. \tag{35}$$

The problem is to find the solution of (35) subject to the boundary conditions

$$d\tilde{P}/dz = 0 \quad \text{on} \quad z = \pm l. \tag{36}$$

Expressing the function $\rho_0(z)\, d\delta(z-z_0)/dz$ as a Fourier series in the range $-l < z-z_0 < l$, we get

$$\rho_0(z)\frac{d\delta}{dz}(z-z_0) = \frac{\rho_0(z_0)}{l}\left[\frac{\beta}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi z}{l}\right) + b_m \sin\left(\frac{m\pi z}{l}\right)\right)\right], \tag{37}$$

where

$$\left.\begin{aligned} a_m &= \frac{m\pi}{l}\sin\left(\frac{m\pi}{l}z_0\right) + \beta\cos\left(\frac{m\pi}{l}z_0\right), \\ b_m &= \beta\sin\left(\frac{m\pi}{l}z_0\right) - \frac{m\pi}{l}\cos\left(\frac{m\pi}{l}z_0\right). \end{aligned}\right\} \tag{38}$$

The solution of (35) satisfying the boundary conditions (36) is given in the elliptic case by

$$\tilde{P} = \frac{N'\rho_0(z_0)}{4\pi l}\left[-\frac{\beta\mu^2}{2\xi_n^2} + \sum_{m=1}^{\infty}\left\{e^{-\frac{1}{2}\beta z}(A_{mn}\cosh k_n z + B_{mn}\sinh k_n z) + \gamma_{mn}\cos\left(\frac{m\pi}{l}z\right) + \delta_{mn}\sin\left(\frac{m\pi}{l}z\right)\right\}\right], \tag{39}$$

where

$$k_n = \left(\frac{\beta^2}{4} + \frac{\xi_n^2}{\mu^2}\right)^{\frac{1}{2}}, \quad \epsilon_{mn} = \left(\frac{\xi_n^2}{\mu^2} + \frac{m^2\pi^2}{l^2}\right)^2 + \beta^2\frac{m^2\pi^2}{l^2}, \tag{40}$$

$$\gamma_{mn} = -\frac{1}{\epsilon_{mn}}\left\{\left(\frac{\xi_n^2}{\mu^2} + \frac{m^2\pi^2}{l^2}\right)a_m + \beta\frac{m\pi}{l}b_m\right\}, \tag{41}$$

$$\delta_{mn} = \frac{1}{\epsilon_{mn}}\left\{\beta\frac{m\pi}{l}a_m - \left(\frac{\xi_n^2}{\mu^2} + \frac{m^2\pi^2}{l^2}\right)bm\right\}, \tag{42}$$

$$A_{mn} = \frac{(-1)^{m+1}2m\pi\mu^2\delta_{mn}}{\xi_n^2 l\sinh 2k_n l}\{k_n\cosh k_n l\sinh\tfrac{1}{2}\beta l + \tfrac{1}{2}\beta\sinh k_n l\cosh\tfrac{1}{2}\beta l\}, \tag{43}$$

$$B_{mn} = \frac{(-1)^{m+1}2m\pi\mu^2\delta_{mn}}{\xi_n^2 l\sinh 2k_n l}\{k_n\sinh k_n l\cosh\tfrac{1}{2}\beta l + \tfrac{1}{2}\beta\cosh k_n l\sinh\tfrac{1}{2}\beta l\}. \tag{44}$$

Using the inversion formula for finite Hankel transforms given by Sneddon (1972, p. 450) and (10), the solution for the pressure p is

$$p = \frac{N'\rho_0(z_0)\,e^{i\omega t}}{2\pi la^2}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r)}{J_0^2(\xi_n a)}\left\{\sum_{m=1}^{\infty}\left[e^{-\frac{1}{2}\beta z}(A_{mn}\cosh k_n z + B_{mn}\sinh k_n z) + \gamma_{mn}\cos\left(\frac{m\pi}{l}z\right) + \delta_{mn}\sin\left(\frac{m\pi}{l}z\right)\right] + \frac{\beta\mu^2}{2\xi_n^2}\right\}. \tag{45}$$

In a similar way the solution for the pressure in the hyperbolic case is obtained as

$$p = \frac{N'\rho_0(z_0)\,e^{i\omega t}}{2\pi la^2}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r)}{J_n^2(\xi_n a)}\left\{\sum_{m=1}^{\infty}\left[e^{-\frac{1}{2}\beta z}(A'_{mn}\cos k'_n z + \beta'_{mn}\sin k'_n z) + \gamma'_{mn}\cos\left(\frac{m\pi}{l}z\right) + \delta'_{mn}\sin\left(\frac{m\pi}{l}z\right)\right] + \frac{\beta\lambda^2}{2\xi_n^2}\right\}, \tag{46}$$

where

$$k'_n = (\xi_n^2/\lambda^2 - \beta^2/4)^{\frac{1}{2}}, \quad \epsilon'_{mn} = \left(\frac{m^2\pi^2}{l^2} - \frac{\xi_n^2}{\lambda^2}\right)^2 + \beta^2\frac{m^2\pi^2}{l^2},$$

$$\gamma'_{mn} = -\frac{1}{\epsilon'_{mn}}\left\{\left(\frac{m^2\pi^2}{l^2} - \frac{\xi_n^2}{\lambda^2}\right)a_m + \beta\frac{m\pi}{l}b_m\right\}, \tag{47}$$

$$\delta'_{mn} = \frac{1}{\epsilon'_{mn}}\left\{\beta\frac{m\pi}{l}a_m - \left(\frac{m^2\pi^2}{l^2} - \frac{\xi_n^2}{\lambda^2}\right)b_m\right\}, \tag{48}$$

$$A'_{mn} = \frac{(-1)^m\, 2m\pi\lambda^2\delta'_{mn}}{\xi_n^2\, l \sin 2k'_n l}(k'_n \cos k'_n l \sinh \tfrac{1}{2}\beta l + \tfrac{1}{2}\beta \sin k'_n l \cosh \tfrac{1}{2}\beta l), \tag{49}$$

$$B'_{mn} = \frac{(-1)^m\, 2m\pi\lambda^2\,\delta'_{mn}}{\xi_n^2\, l \sin 2k'_n l}(-k'_n \sin k'_n l \cosh \tfrac{1}{2}\beta l + \tfrac{1}{2}\beta \cos k'_n l \sinh \tfrac{1}{2}\beta l). \tag{50}$$

The solution for the pressure p in the case of a Boussinesq rotating fluid is given in the elliptic case by

$$p = \frac{N'\rho'_0 e^{i\omega t}}{2a^2l^2}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r)}{J_0^2(\xi_n a)}\left\{\sum_{m=1}^{\infty}\frac{m}{(m^2\pi^2/l^2 + \xi_n^2/\mu^2)}\right.$$
$$\left.\times\left[\frac{(m\pi/l)(-1)^{m+1}\cos(m\pi z_0/l)}{(\xi_n/\mu)\cosh(\xi_n l/\mu)}\sinh\left(\frac{\xi_n}{\lambda}z\right) + \sin\left(\frac{m\pi}{l}(z-z_0)\right)\right]\right\}, \tag{51}$$

and in the hyperbolic case by

$$p = \frac{N'\rho'_0 e^{i\omega t}}{2a^2l^2}\sum_{n=1}^{\infty}\frac{J_0(\xi_n r)}{J_0^2(\xi_n a)}\left\{\sum_{m=1}^{\infty}\frac{m}{(m^2\pi^2/l^2 - \xi_n^2/\lambda^2)}\right.$$
$$\left.\times\left[\frac{(m\pi/l)(-1)^{m+1}\cos(m\pi z_0/l)}{(\xi_n/\lambda)\cos(\xi_n l/\lambda)}\sin\left(\frac{\xi_n}{\lambda}z\right) + \sin\left(\frac{m\pi}{l}(z-z_0)\right)\right]\right\}. \tag{52}$$

The pressure in the limiting case of a homogeneous rotating fluid is obtained by putting $\mu^2 = (\omega^2 - 4\Omega^2)/\omega^2$ and $\lambda^2 = (4\Omega^2 - \omega^2)/\omega^2$ in (51) and (52).

When the point force is acting at the origin, we have $z_0 = 0$ and the solutions (45) and (46) for the pressure for a non-Boussinesq rotating fluid clearly indicate that it will not become zero anywhere in the domain. However, from the solutions for the pressure in a Boussinesq or homogeneous rotating fluid, we observe that it becomes zero at the origin as they involve sine and hyperbolic sine factors only. Similar behaviour, namely vanishing pressure towards the origin, was observed by Baines (1967) in a homogeneous rotating fluid even though the mechanism forcing the oscillations was different. The flow in the hyperbolic case (45) consists of two infinite sets of modes oscillatory with respect to z, one type of mode oscillating with the frequency k'_n and the other with the forcing frequency $m\pi/l$. The amplitude of the mode with frequency k'_n becomes infinite (resonance occurs) when $\sin 2k'_n l = 0$, that is, when $k'_n l = m\pi$ or $\frac{1}{2}(2m+1)\pi$.

In the limit $\omega \to 2\Omega$, the pressure becomes zero for non-Boussinesq, Boussinesq and homogeneous rotating fluids. The pressure becomes infinite as $\omega \to N$ in non-Boussinesq and homogeneous rotating fluids, but in the case of a Boussinesq rotating fluid we get

$$p = \frac{N'\rho_0' e^{i\omega t}}{2\pi^2 a^2} \sum_{n=1}^{\infty} \frac{J_0(\xi_n r)}{J_0^2(\xi na)} \left\{ \sum_{m=1}^{\infty} \frac{1}{m} \left[(-1)^{m+1} \frac{m\pi}{l} \cos\left(\frac{m\pi}{l} z_0\right) + \sin\left(\frac{m\pi}{l} z - z_0\right) \right] \right\}. \quad (53)$$

Thus the exact resonance existing in flows of non-Boussinesq and homogeneous rotating fluids as $\omega \to N$ is avoided in a Boussinesq rotating fluid.

Solution (46) for the pressure is rewritten in the form

$$p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \exp\left(i\omega t - \tfrac{1}{2}\beta z\right) \frac{J_0(\xi_n r)}{J_0^2(\xi_n a)} \{a_{mn} \cos[k_n'(z - z_0)] + b_{mn} \sin[k_n'(z - z_0)]\} \quad (54)$$

+ modes oscillatory in z with frequencies $m\pi/l$, where a_{mn} and b_{mn} are some known constants. For large ξ_n and n we have

$$\xi na = (n - n_0 + \tfrac{1}{4})\pi - 3/8\pi n + O(n^{-2}), \quad (55)$$

where n_0 is an integer. Using the asymptotic form for $J_0(\xi_n r)$ we may write

$$p \sim \text{analytic terms} + \frac{h_1 e^{i\omega t}}{r^{\frac{1}{2}}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\xi_n r - \frac{1}{4}\pi)}{an^{\frac{7}{2}}}$$
$$\times \left[\frac{h_2 \cos k_n'(z - z_0)}{\sin k_n' l} + \frac{h_3 \sin[k_n'(z - z_0)]}{\cos k_n' l} \right] \left(1 + O\left(\frac{1}{n}\right)\right), \quad (56)$$

where h_1, h_2 and h_3 are constants.

An inspection of the series reveals that the third derivatives of p are discontinuous (for more detailed analysis of similar cases see Wood 1965; Baines 1967). These discontinuities will be realized in the second derivatives of the velocity across the cones $\lambda r \pm (z - z_0) = \text{constant}$, which emanate from the point of application of the force, and continue by reflexion at the lateral side of the cylinder (Oser 1958). These cones may be recognized as the well-known characteristic surfaces, whose presence is explained by the hyperbolic nature of the governing equations. For Boussinesq or homogeneous rotating fluids it is observed that the terms in the infinite series are $O(n^{-\frac{5}{2}})$ and the discontinuities will be realized in the second derivatives of p, i.e. in the velocity gradients instead of the second derivatives of the velocity.

The author is extremely grateful to Prof. Sir James Lighthill for his valuable suggestions for improvements to an earlier version of the paper and for his kind encouragement.

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